Problem sheet M1 IPP

Here is a non-exhaustive list of problems that a student enrolled in the M1 should be able to solve without any trouble before the beginning of the school year. We strongly advise prospective students to check that they have the right skills when they decide to apply.

Algebra (mostly linear algebra)

Problem 1

Let X, Y two sets, $f : X \to Y$.

1. Remind the definition of $f(A)$ and $f^{-1}(B)$, for $A \subseteq X$, $B \subseteq Y$.
2. Show that for all $B \subseteq Y$ $f^{-1}(B^{\complement}) = (f^{-1}(B))^{\complement}$. Do we have $f(A^{\complement}) = (f(A))^{\complement}$ in general?
3. Show that for a collection $(A_i)_{i\in I}$, resp. $(B_i)_{i\in I}$ of subsets of X, resp. of Y,

$$f\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f(A_i),\ f^{-1}\left(\bigcup_{i\in I} B_i\right) = \bigcup_{i\in I} f^{-1}(B_i),\ f^{-1}\left(\bigcap_{i\in I} B_i\right) = \bigcap_{i\in I} f^{-1}(B_i).$$

Do we have $f\left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I} f(A_i)$ in general?

Problem 2

Which is the largest number of linearly independent vectors in $\mathbb{R}^n$?

Problem 3

Show that a symmetric real matrix is positive semidefinite if and only if all its eigenvalues are nonnegative.

Problem 4

Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}$. Find all the eigenvalues of A and an orthonormal basis of eigenvectors. Then, compute A^n, for all integers $n \geq 1$.

Problem 5

1. Prove that any rank 1 matrix $A \in \mathbb{R}^{n\times p}$ is of the form $A = uv^\top$, for some $u \in \mathbb{R}^n \setminus \{0\}, v \in \mathbb{R}^p \setminus \{0\}$ $(n, p \geq 1)$.
2. Prove that for any rank 1 symmetric matrix $A \in \mathbb{R}^{n\times n}$, there is a vector $u \in \mathbb{R}^n \setminus \{0\}$ such that either $A = uu^\top$, or $A = -uu^\top$. In each case, compute the eigenvalues of A and determine their corresponding eigenspaces.

Problem 6

Let $A \in \mathbb{R}^{n\times p}$ and $B \in \mathbb{R}^{p\times n}$ (n and p are two positive integers). Show that AB and BA have the same (possibly complex) eigenvalues.

Problem 7

Prove the following statements, where n, p, d are positive integers.

1. For all $A, B \in \mathbb{R}^{n\times p}$, $\operatorname{rank}(A + B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$.
2. For all $A \in \mathbb{R}^{n\times d}$, $A^\top A$ is invertible if and only if $\operatorname{rank}(A) = d$.
3. For all $A \in \mathbb{R}^{n\times p}$ and $B \in \mathbb{R}^{p\times d}$, $\operatorname{rank}(AB) \leq \min(\operatorname{rank}(A), \operatorname{rank}(B))$.
4. For all $A \in \mathbb{R}^{n\times d}$ and $B \in \mathbb{R}^{d\times n}$, $\operatorname{rank}(AB) = \operatorname{rank}(BA)$.

Calculus

Problem 8

Show that the function

$$x \in \mathbb{R}^n \mapsto \frac{1}{2}\|x\|^2$$

is continuously differentiable.

Problem 9

Show that the function

$$x \in \mathbb{R} \mapsto \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is differentiable, but its derivative is not continuous at 0.

Problem 10

If all $x \in \mathbb{R}^n$ are local minimizers of a differentiable function defined on $\mathbb{R}^n$, is this surely constant? What if the function is assumed to be only continuous (but not necessarily differentiable)?

Problem 11

Let $f, g : \mathbb{R} \to \mathbb{R}$ two C^1 functions. We define $F(x) := \int_0^{g(x)} f(t)\,\mathrm{d}t$, $x \in \mathbb{R}$. Show that F is C^1 and that $F'(x) = f(g(x))\,g'(x)$, for all $x \in \mathbb{R}$.

Problem 12

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a twice differentiable function and $x, y \in \mathbb{R}^d$ be fixed vectors. For $t \in \mathbb{R}$, let $\phi(t) = f((1-t)x + ty)$. Compute $\phi'(t)$ and $\phi''(t)$, for any $t \in \mathbb{R}$, using the gradient and the Hessian of f.

Problem 13

Let $T = \{(x, y) \in \mathbb{R}^2 : 0 \leq x, 0 \leq y, x+y \leq 1\}$. Compute $\int_T f$, where $f(x, y) = x + e^y$.

Problem 14

Let $T = \{(x_1, \ldots, x_d) \in (\mathbb{R}_+)^d : a_1x_1 + \ldots + a_dx_d \leq 1\}$, where $a_1, \ldots, a_d > 0$ and $d \geq 1$ is a positive integer. For all $i = 1, \ldots, d$, compute $\int_T x_i\,\mathrm{d}x_1 \ldots \mathrm{d}x_d$ and $\int_T x_i^2\,\mathrm{d}x_1 \ldots \mathrm{d}x_d$.

Real analysis

Problem 15

1. Is the union of two convex sets convex? What about their intersection?
2. Is the union of two closed subsets of $\mathbb{R}^n$ always closed? What about their Minkowski sum?
3. Is there a dense convex subset U of $\mathbb{R}^3$ such that $(-1, 2, 7)^\top \notin U$?

Problem 16

Give an example of a

1. nonconvex function;

2. nondifferentiable function.

Problem 17

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix and $f(x) = x^\top Ax$, for all $x \in \mathbb{R}^n$. Prove that f is convex if and only if A is positive semidefinite.

Problem 18

Prove that the following functions are convex:

1. $f(x) = \|x\|, x \in \mathbb{R}^d$, where $\|\cdot\|$ is any norm in $\mathbb{R}^d$;
2. $f(x) = \sqrt{x^\top Ax}, x \in \mathbb{R}^d$, where $A \in \mathbb{R}^{d\times d}$ is a symmetric positive semidefinite matrix;
3. $f(x) = \|x - \pi_E(x)\|_2, x \in \mathbb{R}^d$, where $\|\cdot\|_2$ is the Euclidean norm, $E \subseteq \mathbb{R}^d$ is a closed, convex set and π_E is the metric projection on E, i.e., $\pi_E(x)$ is the closed point in E to x.

Problem 19

1. Let $A \subseteq \mathbb{R}$ be nonempty. Remind the definition of $\inf A$ (in $\mathbb{R}\cup\{-\infty\}$), and $\sup A$ (in $\mathbb{R}\cup\{+\infty\}$). What is the difference with $\min A$ and $\max A$ (when they exist) ?

2. Show that there exists sequences (u_n), (v_n) of elements of A such that

$$\lim_{n\to+\infty} u_n = \inf A, \quad \lim_{n\to+\infty} v_n = \sup A$$

Problem 20

Let $(u_n)_{n\geq 1}$ be a sequence of real numbers. We remind the definitions:

$$\limsup_{n\to\infty} u_n := \lim_{n\to\infty} \sup_{k\geq n} u_k, \quad \liminf_{n\to\infty} u_n := \lim_{n\to\infty} \inf_{k\geq n} u_k$$

Show that $\limsup\limits_{n\to\infty} u_n$ and $\liminf\limits_{n\to\infty} u_n$ are always well-defined in $\mathbb{R}\cup\{\pm\infty\}$, and when they are finite, they are respectively the highest and lowest accumulation points of $(u_n)_{n\geq 1}$.

Problem 21

Let $f_n(x) := x^n$, $n \in \mathbb{N}$, $x \in [0, 1]$. Show that f_n converges pointwise to the function $f(x) = 1_{\{1\}}(x)$ on $[0, 1]$, but not uniformly on $[0, 1]$.

Problem 22

Let $(f_n)_{n\geq 1}$ be a sequence of real valued functions defined on some interval $I \subseteq \mathbb{R}$. Assume that f_n converges pointwise to a function f on I.

1. Do the following propositions hold true ?
   - if all the f_n are non-decreasing, so is f.
   - if all the f_n are increasing, so is f.
   - if all the f_n are T-periodic, so is f.
   - if all the f_n are continuous at a point $a \in I$, so is f.
2. Same question replacing the assumption of pointwise convergence by uniform convergence.

Probability theory

Problem 23

Let X and Y be i.i.d. exponential random variables (i.e., they have the exponential distribution with some parameter $\lambda > 0$). Prove that $\min(X, Y)$ and $|X - Y|$ are independent.

Problem 24

Let R have the exponential distribution with parameter $1/2$ and independently let Θ have the uniform distribution on $[0, 2\pi]$. Find the law of the pair $(\sqrt{R}\cos\Theta, \sqrt{R}\sin\Theta)$.

Problem 25

Recall that the Gamma distribution with parameters $\alpha > 0$ and $\lambda > 0$ is the law on the positive real numbers with density

$$f_{\alpha,\lambda}\colon x \mapsto \frac{1}{\Gamma(\alpha)}\lambda^{\alpha}x^{\alpha-1}\mathrm{e}^{-\lambda x}1_{x>0}.$$

Let X and Y be independent random variables, with the Gamma distribution with parameters (α, λ) and (β, λ) respectively.

1. Find a joint density for $U = X + Y$ and $V = X/(X + Y)$.
2. What are the marginal distributions of U and V? Are they independent?

3. Show that $\mathbb{E}\Big[\dfrac{X}{X+Y}\Big] = \dfrac{\mathbb{E}[X]}{\mathbb{E}[X]+\mathbb{E}[Y]}$. Does this hold for general random variables?

Problem 26

Let $c, \lambda > 0$ and consider the following density on $\mathbb{R}^2$:

$$f_{(X,Y)} \colon (x,y) \mapsto c\,\mathrm{e}^{-\lambda x} 1_{0<y<x}.$$

Let us denote by (X,Y) a random pair with this law.

1. Find the value of c.
2. What is the respective law of X and Y? Are they independent?
3. Find the law of $X - Y$.
4. Compute $\mathbb{E}[XY]$.
5. Compute $\mathbb{E}[X|Y]$.
6. Compute $\mathbb{E}[Y|X]$.

Problem 27

Let X and Y be independent random variables with the Poisson distribution with parameters $\theta_1 > 0$ and $\theta_2 > 0$ respectively.

1. Compute the moment generating function $\mathbb{E}[s^{X+Y}]$ for every $s \in [0,1]$. What is the law of $X+Y$?
2. Compute the conditional moment generating function $\mathbb{E}[s^X|X+Y]$ for every $s \in [0,1]$ and deduce the conditional law of X given $X+Y$.

Problem 28

For all positive integers n, let X_n be a binomial random variable with parameters n and λ/n, where $\lambda > 0$ is fixed. Prove that X_n converges in distribution to the Poisson distribution with parameter λ.

Problem 29

For all positive integers n, let X_n be a random variable with the Poisson distribution with parameter $1/n$. Prove that $e^{n!}X_n$ converges in probability to 0.

Problem 30

Let $(X_n)_{n\geq 1}$ be a sequence of real random variables. Show that X_n converges in probability to zero if and only if $\mathbb{E}\left[\frac{|X_n|}{|X_n|+1}\right] \xrightarrow[n\to\infty]{} 0$.

Problem 31

Let $X_1, X_2, \ldots$ be i.i.d. real valued random variables. For all positive integers n, let Y_n be the number of indices $i \in \{1, \ldots, n\}$ such that $X_{2i-1} < X_{2i}$. Prove that Y_n/n converges almost surely.

Problem 32

Let $X_1, X_2, \ldots$ be i.i.d. real valued random variables. Assume that $\mathbb{E}[X_1^2]$ is finite. For all positive integers n, let V_n be the empirical variance of $X_1, \ldots, X_n$. Prove that V_n converges in probability to the variance of X_1.

Problem 33

Let $X_1, X_2, \ldots$ be i.i.d. uniform random variables in $[0, 1]$ For all positive integers n, let $M_n = \min(X_1, \ldots, X_n)$.

1. Compute the cumulative distribution function of M_n.
2. Find the expectation of M_n.
3. Show that M_n converges in probability to zero.
4. Show that M_n converges almost surely to zero.
5. Show that nM_n converges in distribution and find the limit.

Problem 34

Let $X_1, X_2, \ldots$ be i.i.d. real-valued random variables with finite variance. Study the convergence (and specify in what sense) of the following sequences as $n \to \infty$:

1. $\dfrac{X_1^2 + \cdots + X_n^2}{n}$.
2. $\dfrac{X_1X_2 + X_3X_4 + \cdots + X_{2n-3}X_{2n-2} + X_{2n-1}X_{2n}}{n}$.
3. $\dfrac{X_1X_2 + X_2X_3 + \cdots + X_{2n-2}X_{2n-1} + X_{2n-1}X_{2n}}{n}$.

Problem 35

Let $X_1, X_2, \ldots$ be i.i.d. real-valued random variables with mean $m \in \mathbb{R}$ and finite variance $\sigma^2 > 0$. For every $n \geq 1$, let us set:

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad \text{and} \qquad \widehat{\sigma}_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2.$$

1. Compute the expectation and variance of $\overline{X}_n$.
2. Prove that $\sqrt{n} \cdot \dfrac{\overline{X}_n - m}{\sigma}$ converges in distribution and express its limit.
3. Compute the expectation of $\widehat{\sigma}_n^2$.
4. In which sense does $\widehat{\sigma}_n^2$ converge? Express the limit.
5. Prove that $\sqrt{n} \cdot \dfrac{\overline{X}_n - m}{\widehat{\sigma}_n}$ converges in distribution to a standard Gaussian law.

Problem 36

Let $X_1, X_2, \ldots$ be i.i.d. real-valued random variables with zero mean and finite variance. Let $Z_n = n^{-1/2}(X_1 + \cdots + X_n)$ for every $n \geq 1$.

1. Find the limit in distribution of Z_n as $n \to \infty$.
2. Show that there exist $a, b \in \mathbb{R}$ such that for every $n \geq 1$, we have $Z_{2n} - Z_n = aZ_n + bZ_n'$, where Z_n' has the same law as Z_n and is independent from it.
3. Conclude that Z_n does not converge in probability.

Problem 37

Given two i.i.d. random vectors X, Y with the d-dimensional standard normal distribution, show that $X + Y$ and $X - Y$ are independent ($d \geq 1$).

Problem 38

Let X and Y be real random vectors in $\mathbb{R}^p$ and $\mathbb{R}^n$ respectively ($n, p \geq 1$) and assume that their joint distribution is Gaussian with mean $\mu \in \mathbb{R}^{p+n}$ and covariance matrix $\Sigma \in \mathbb{R}^{(p+n)\times(p+n)}$. For all $x \in \mathbb{R}^p$, find the conditional distribution of Y given $X = x$.